Lecture 20

Ex: Find the tangent plane to the surface parametrized by $\vec{r}(u,v) = \langle u^2 - v^2, u+v, u^2+3v \rangle$ at the point $(3,1,1)$.

Sol: We begin by finding the u_0 & v_0 such that

$$\vec{r}(u_0,v_0) = \langle 3,1,1 \rangle : \begin{cases} u^2 - v^2 = 3 & (1) \\ u+v = 1 & (2) \\ u^2+3v = 1 & (3) \end{cases}$$

$$(1): u^2 - v^2 = \underbrace{(u+v)}_{(2)}(u-v) = (1)(u-v) = u-v = 3 \Rightarrow u = 3+v$$

$$(2): u+v = (3+v)+v = 3+2v = 1 \Rightarrow v = -1 \Rightarrow u = 2$$

Check $(u_0,v_0) = (2,-1)$ w/ (3): $2^2 + 3(-1) = 4-3 = 1$ ✓

So, $\vec{r}(2,-1) = \langle 3,1,1 \rangle$.

$$\frac{\partial \vec{r}}{\partial u} = \langle 2u, 1, 2u \rangle, \quad \frac{\partial \vec{r}}{\partial u}(2,-1) = \langle 4,1,4 \rangle = \vec{r}_u$$

$$\frac{\partial \vec{r}}{\partial v} = \langle -2v, 1, 3 \rangle, \quad \frac{\partial \vec{r}}{\partial v}(2,-1) = \langle 2,1,3 \rangle = \vec{r}_v$$

Tangent plane is parametrized by:

$$T_{(3,1,1)}S(s,t) = \langle 3,1,1 \rangle + s\langle 4,1,4 \rangle + t\langle 2,1,3 \rangle$$
$$= \langle 3+4s+2t, 1+s+t, 1+4s+3t \rangle$$

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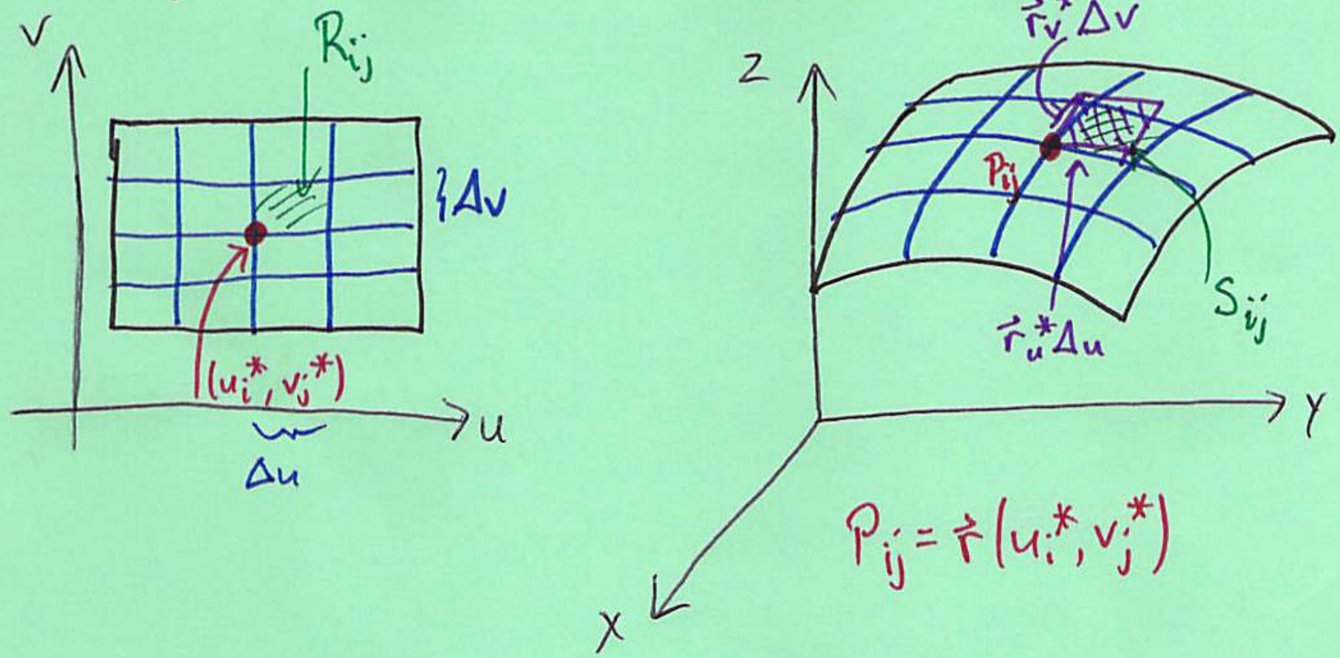
We can now ask about finding surface areas of surfaces. Suppose we have

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle, \quad (u,v) \in D$$

to approximate the area, we'll "tile" it with flat parallelograms, then let them get smaller



$$P_{ij} = \vec{r}(u_i^*, v_j^*)$$

$$\Delta S_{ij} = \text{area of parallelogram over } S_{ij}$$

$$= \|\vec{r}_u^{\,*} \times \vec{r}_v^{\,*}\| \Delta u \Delta v$$

This gives the surface area of S as:

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta S_{ij} =: \iint_S dS = \iint_D \|\vec{r}_u \times \vec{r}_v\| \, dA$$

($dA = du\,dv$ or $dv\,du$)

So, $\boxed{dS = \|\vec{r}_u \times \vec{r}_v\| \, dA}$

Ex: Find the surface area of the cylinder described by $x^2+y^2=4,\ 0\leq z\leq 3$.

Sol: Begin by parametrizing the surface. In cylindrical, it's described by $r=2,\ 0\leq z\leq 3$. So,

$$\vec{r}(\theta,z)=\langle 2\cos\theta, 2\sin\theta, z\rangle,\ \underbrace{0\leq\theta\leq 2\pi,\ 0\leq z\leq 3}_{D}$$

$$\vec{r}_\theta=\langle -2\sin\theta, 2\cos\theta, 0\rangle,\quad \vec{r}_z=\langle 0,0,1\rangle$$

$$\vec{r}_\theta\times\vec{r}_z=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2\sin\theta & 2\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=\langle 2\cos\theta, 2\sin\theta, 0\rangle$$

$$\|\vec{r}_\theta\times\vec{r}_z\|=\sqrt{4\cos^2\theta+4\sin^2\theta+0}=\sqrt{4}=2$$

So,

$$A(S)=\iint_S dS=\iint_D\|\vec{r}_\theta\times\vec{r}_z\|\,dA=\int_0^{2\pi}\int_0^3 2\,dz\,d\theta=12\pi.$$

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16.7 - Surface Integrals

Def: The surface integral of f over S (or scalar surface integral) is given by

$$\boxed{\iint_S f\,dS=\iint_D f(\vec{r}(u,v))\,\|\vec{r}_u\times\vec{r}_v\|\,dA}$$

Ex: Compute the surface integral $\iint_S xyz\,dS$ where S is the piece of the cone $z^2 = x^2+y^2$ in the first octant, below $z=1$.

Sol: First, parametrize the surface:

Using cylindrical:

$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$, $0 \le r \le 1$, $0 \le \theta \le \frac{\pi}{2}$.

$\vec{r}_r = \langle \cos\theta, \sin\theta, 1\rangle$, $\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r\rangle$$

$$\|\vec{r}_r \times \vec{r}_\theta\| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}\,r$$

$$\iint_S xyz\,dS = \int_0^{\frac{\pi}{2}}\int_0^1 r^3\cos\theta\sin\theta(\sqrt{2}r)\,dr\,d\theta = \frac{\sqrt{2}}{5}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta$$

$$= \frac{\sqrt{2}}{5}\left(\frac{1}{2}\sin^2\theta\right)\Big|_0^{\frac{\pi}{2}} = \frac{\sqrt{2}}{10}$$

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An Application

If the surface S has density $\rho(x,y,z)$, the <u>mass</u> of S is $m = \iint_S \rho\,dS$ & the center of mass is

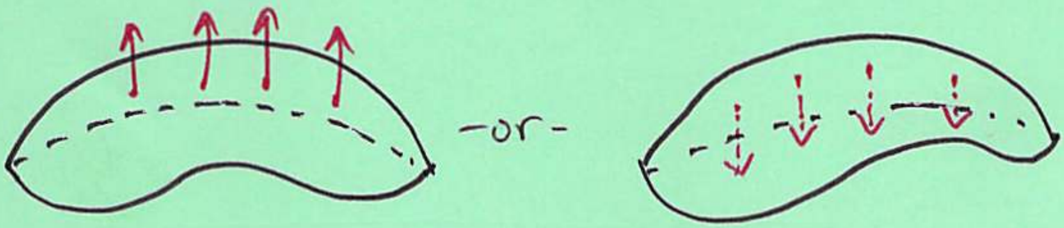
$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{1}{m}\iint_S x\rho\,dS, \frac{1}{m}\iint_S y\rho\,dS, \frac{1}{m}\iint_S z\rho\,dS\right)$$

To continue with the next type of integral (also, the last!) we need to talk about the orientation of a surface.

Def: An orientation on a surface S is a choice of a continuous unit normal vector field on S.

Fact: If a surface is orientable, it has exactly two orientations!

Ex:



If the two orientations are $\vec{n}_1$ & $\vec{n}_2$, then

$$\vec{n}_1 = -\vec{n}_2.$$

If S is parametrized by $\vec{r}(u,v)$, then:

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \quad \& \quad -\vec{n} = \frac{\vec{r}_v \times \vec{r}_u}{\|\vec{r}_v \times \vec{r}_u\|}$$

are the two choices of orientation.

Def: A surface which is the boundary of a solid is called a closed surface. On a closed surface, the positive orientation is the outward one.

Ex: Find the upward pointing orientation on the surface which is the graph of $f(x,y)=x^2+y^2$, over $x^2+y^2\le 9$.

Sol:

First, parametrize the surface:

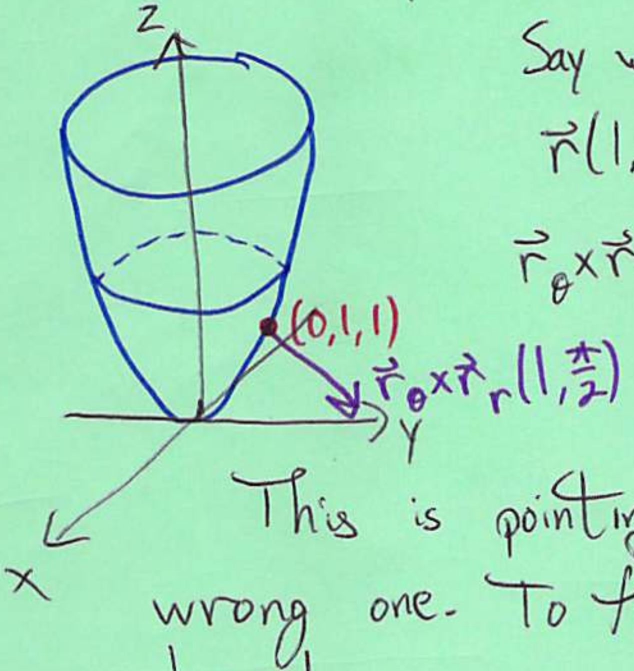
$$\vec{r}(r,\theta)=\langle r\cos\theta, r\sin\theta, r^2\rangle, \quad 0\le r\le 3,\ 0\le\theta\le 2\pi.$$

Then $\vec{r}_r=\langle \cos\theta, \sin\theta, 2r\rangle$, $\vec{r}_\theta=\langle -r\sin\theta, r\cos\theta, 0\rangle$

$$\vec{r}_\theta\times\vec{r}_r=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2r \end{vmatrix}=\langle 2r^2\cos\theta, 2r^2\sin\theta, -r\rangle$$

$$\|\vec{r}_\theta\times\vec{r}_r\|=\sqrt{4r^4+r^2}=r\sqrt{4r^2+1}$$

Was $\vec{r}_\theta\times\vec{r}_r$ the right order? To check, we check its direction at a point on the surface:



Say we take $(r,\theta)=(1,\frac{\pi}{2})$

$\vec{r}(1,\frac{\pi}{2})=\langle 0,1,1\rangle$

$\vec{r}_\theta\times\vec{r}_r(1,\frac{\pi}{2})=\langle 0,2,-1\rangle$

This is pointing downward, so we chose the wrong one. To fix this, switch the order, i.e., multiply by -1.

So, $\vec{n}$ is:

$$\vec{n} = \frac{\vec{r}_r \times \vec{r}_\theta}{\|\vec{r}_r \times \vec{r}_\theta\|} = \frac{\langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle}{r\sqrt{4r^2+1}}$$

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Lecture 21

Flux:

The concept of flux is measuring the rate at which something flows through a surface (e.g. air through a butterfly net).

$$d\vec{S} = \vec{n}\,dS = \frac{\vec{r}_u \times \vec{r}_v}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}}\left(\cancel{\|\vec{r}_u \times \vec{r}_v\|}\,dA\right) = (\vec{r}_u \times \vec{r}_v)\,dA$$

Def: If $\vec{F}$ is a continuous vector field defined on a surface S which has orientation $\vec{n}$, then the flux of $\vec{F}$ across S is

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_S (\vec{F}\cdot\vec{n})\,dS = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\,dA}$$

[S parametrized by $\vec{r}(u,v)$, $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$]